Double Domination Number and Connectivity of Graphs

C. Sivagnanam
Department of Information Technology
Al Musanna College of Technology
Sultanate of Oman
E-mail: choshi71@gmail.com

ABSTRACT

In a graph G, a vertex dominates itself and its neighbours. A subset S of V is called a dominating set in G if every vertex in V is dominated by at least one vertex in S. The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set. A set \( S \subseteq V \) is called a double dominating set of a graph G if every vertex in V is dominated by at least two vertices in S. The minimum cardinality of a double dominating set is called double domination number of G and is denoted by \( dd(G) \). The connectivity \( \kappa(G) \) of a connected graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. In this paper we find an upper bound for the sum of the double domination number and connectivity of a graph and characterize the corresponding extremal graphs.

Key Words: Domination, Domination number, Double domination, double domination number and Connectivity.

1. INTRODUCTION

The graph G = (V,E) we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of G are denoted by m and n respectively. The degree of any vertex u in G is the number of edges incident with u and is denoted by \( d(u) \). The minimum and maximum degree of a graph G is denoted by \( \delta(G) \) and \( \Delta(G) \) respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1] and Haynes et al [2, 3].

Let \( v \in V \). The open neighbourhood and closed neighbourhood of \( v \) are denoted by \( N(v) \) and for all \( v \in S \) and \( N[S] = N(S) \cup S \). If \( S \subseteq V \) then \( N(S) = \bigcup_{v \in S} N(v) \).

\( N[v] = N(v) \cup \{v\} \). If \( S \subseteq V \) and \( u \in S \) then the private neighbour set of u with respect to S is defined by \( pn[u,S] = \{v : N[v] \cap S = \{u\}\} \). If \( \{m_1,m_2,\ldots,m_n\} \) denotes the graph obtained from the graph H by pasting \( m_i \) edges to the vertex \( v_i \in V(H) \), 1 \( \leq i \leq n \). H\((P_{m_1},P_{m_2},\ldots,P_{m_n})\) is the graph obtained from the graph H by attaching the end vertex of \( P_{m_i} \) to the vertex \( v_i \) in H , 1 \( \leq i \leq n \). Bistar \( B(r,s) \) is a graph obtained from \( K_{1,r} \) and \( K_{1,s} \) by joining its centre vertices by an edge.

In a graph G, a vertex dominates itself and its neighbours. A subset S of V is called a dominating set in G if every vertex in V is dominated by at least one vertex in
The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. Harary and Haynes [4] introduced the concept of double domination in graphs. A set $S \subseteq V$ is called a double dominating set of a graph $G$ if every vertex in $V$ is dominated by at least two vertices in $S$. The minimum cardinality of double dominating set is called double domination number of $G$ and is denoted by $dd(G)$. The connectivity $\kappa(G)$ of a connected graph $G$ is the minimum number of vertices whose removal results in a disconnected or trivial graph.

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. J.Paulraj Joseph and S.Arumugam [5] proved that $dd(G) \leq n$. We use the following theorems.

**Theorem 1.1.** [2] For any graph $G$, $dd(G) \leq n$.

**Theorem 1.2.** [1] For a graph $G$, $\kappa(G) \leq \delta(G)$.

### 2. MAIN RESULTS

**Theorem 2.1.** For any connected graph $G$ $dd(G) + \kappa(G) \leq 2n - 1$ and equality holds if and only if $G$ is isomorphic to $K_2$.

**Proof.** $dd(G) + \kappa(G) \leq n + \delta \leq n + n - 1 = 2n - 1$.

Let $dd(G) + \kappa(G) = 2n - 1$. Then $dd(G) = n$ and $\kappa(G) = n - 1$. Then $G$ is a complete graph on $n$ vertices. Since $dd(K_n) = 2$ we have $n = 2$. Hence $G$ is isomorphic to $K_2$. The converse is obvious.

**Theorem 2.2.** For any connected graph $G$, $dd(G) + \kappa(G) = 2n - 2$ if and only if $G$ is isomorphic to $K_3$ or $K_{1,2}$.

**Proof.** Let $dd(G) + \kappa(G) = 2n - 2$. Then there are two cases to consider.

(i) $dd(G) = n - 1$ and $\kappa(G) = n - 1$

(ii) $dd(G) = n$ and $\kappa(G) = n - 2$

**Case 1.** $dd(G) = n - 1$ and $\kappa(G) = n - 1$

Then $G$ is a complete graph on $n$ vertices. Since $dd(K_n) = 2$ we have $n = 3$. Hence $G$ is isomorphic to $K_3$.

**Case 2.** $dd(G) = n$ and $\kappa(G) = n - 2$

Then $n - 2 \leq \delta(G)$. If $\delta = n - 1$ then $G$ is a complete graph which is a contradiction. Hence $\delta = n - 2$. Then $G$ is isomorphic to $K_n - Y$ where $Y$ is a matching in $K_n$. Then $dd(G) = 2$ or $3$. Since $dd(G) = 2$ is isomorphic we have $n = 3$. Hence $G$ is isomorphic to $K_{1,2}$. The converse is obvious.

**Theorem 2.3.** For any connected graph $G$ $dd(G) + \kappa(G) = 2n - 3$ if and only if $G$ is isomorphic to $K_4$ or $C_4$ or $P_4$ or $K_{1,3}$. 


Proof. Let \( \dd(G) + \kappa(G) = 2n - 3 \). Then there are three cases to consider
(i) \( \dd(G) = n - 2 \) and \( \kappa(G) = n - 1 \)
(ii) \( \dd(G) = n - 1 \) and \( \kappa(G) = n - 2 \)
(iii) \( \dd(G) = n \) and \( \kappa(G) = n - 3 \)

Case 1. \( \dd(G) = n - 2 \) and \( \kappa(G) = n - 1 \)
Then G is a complete graph on \( n \) vertices. Since \( \dd(K_n) = 2 \) we have \( n = 4 \). Hence G is isomorphic to \( K_4 \).

Case 2. \( \dd(G) = n - 1 \) and \( \kappa(G) = n - 2 \)
Then \( n - 2 \leq \delta(G) \). If \( \delta = n - 1 \) then G is a complete graph which gives a contradiction. Hence \( \delta(G) = n - 2 \). Then G is isomorphic to \( K_n - Y \) where \( Y \) is a matching in \( K_n \). Then \( \dd(G) = 2 \) or 3. If \( \dd(G) = 2 \) then \( n = 3 \). Hence G is isomorphic to \( K_{1,2} \) which is a contradiction. Thus \( \dd(G) = 3 \). Then \( n = 4 \) and hence G is isomorphic to \( C_4 \).

Case 3. \( \dd(G) = n \) and \( \kappa(G) = n - 3 \)
Then \( n - 3 \leq \delta \). If \( \delta = n - 1 \) then G is a complete graph which is a contradiction. If \( \delta = n - 2 \) then G is isomorphic to \( K_n - Y \) where \( Y \) is a matching in \( K_n \). Then \( \dd(G) = 2 \) or 3. Then \( n = 2 \) or 3 which is a contradiction to \( \kappa(G) = n - 3 \). Hence \( \delta = n - 3 \). Let \( X \) be vertex cut of G with \( |X| = n - 3 \) and let \( V - X = \{x_1, x_2, x_3\} \), \( X = \{v_1, v_2, v_3, \ldots, v_{n-3}\} \).

Sub case 3.1. \( \langle V - X \rangle = \overline{K_3} \)
Then every vertex of \( V \) - \( X \) is adjacent to all the vertices of \( X \). Then \( \langle V - X \rangle \cup \{v_1\} \) is a double dominating set of G hence \( \dd(G) \leq 4 \). This gives \( n \leq 4 \). Since \( n \leq 3 \) is impossible, we have \( n = 4 \). Hence G is isomorphic to \( K_{1,3} \).

Sub case 3.2. \( \langle V - X \rangle = K_1 \cup K_2 \)
Let \( x_1, x_2 \in E(G) \). Then \( x_3 \) is adjacent to all the vertices in \( X \) and \( x_1, x_2 \) are not adjacent to at most one vertex in \( X \).

If \( v_1 \notin N(x_1) \cup N(x_2) \) then \( \langle V - X \rangle \cup \{v_1\} \) is a double dominating set of G and hence \( \dd(G) \leq 4 \). This gives \( n = 4 \) which is a contradiction to G is a connected graph. So all \( v_i \in \) either \( N(x_1) \) or \( N(x_2) \) or both.
Then \( \langle V - X \rangle \cup \{v_i\} \) is a double dominating set of G. Hence \( \dd(G) \leq 4 \) and then \( n = 4 \).
Then G is isomorphic to \( P_4 \) or \( K_4(1,0,0) \). But \( \dd(K_4(1,0,0)) = 3 \neq n \) which is a contradiction. The converse is obvious. □

Theorem 2.4. For any connected graph G \( \dd(G) + \kappa(G) = 2n - 4 \) if and only if G is isomorphic to \( K_4 - e \) or \( K_{1,4} \) or \( K_3 \) (1,0,0) or \( B(1,2) \) or \( K_5 - Y \) where \( Y \) is a matching on \( K_5 \) with \( |Y| = 2 \).

Proof. Let \( \dd(G) + \kappa(G) = 2n - 4 \). Then there are four cases to consider
(i) \( \dd(G) = n - 3 \) and \( \kappa(G) = n - 1 \)
(ii) \( \dd(G) = n - 2 \) and \( \kappa(G) = n - 2 \)
(iii) \( \dd(G) = n - 1 \) and \( \kappa(G) = n - 3 \)
(iv) \( \dd(G) = n \) and \( \kappa(G) = n - 4 \)

Case 1. \( \dd(G) = n - 3 \) and \( \kappa(G) = n - 1 \)
Then G is a complete graph on \( n \) vertices. Since \( \dd(G) = 2 \) we have \( n = 5 \). Hence G is isomorphic to \( K_5 \).

Case 2. \( \dd(G) = n - 2 \) and \( \kappa(G) = n - 2 \)
Then $n-2 \leq \delta$. If $\delta = n-1$ then $G$ is a complete graph which is a contradiction. If $\delta(G) = n-2$ then $G$ is isomorphic to $K_{n-Y}$ where $Y$ is a matching in $K_n$. Then $dd(G) = 2$ or 3. If $dd(G) = 2$, then $n = 4$ then $G$ is either $C_4$ or $K_{4-e}$. But $dd(C_4) = 3 \neq n-2$. Hence $G$ is isomorphic to $K_{4-e}$. If $dd(G) = 3$ then $n = 5$ and hence $G$ is isomorphic to $K_{5-Y}$ where $Y$ is a matching on $K_5$ with $|Y| = 2$.

**Case 3.** $dd(G) = n-1$ and $\kappa(G) = n-3$

Then $n-3 \leq \delta$. If $\delta = n-1$ then $G$ is a complete graph which is a contradiction. If $\delta = n-2$ then $G$ is isomorphic to $K_{n-Y}$ where $Y$ is a matching in $K_n$. Then $dd(G) = 2$ or 3. Then $n = 3$ or 4. Since $n = 3$ is impossible, we have $n = 4$. Then $G$ is either $K_4$ or $C_4$. For these two graphs $\kappa(G) = n-3$ which is a contradiction.

Hence $\delta = n-3$.

Let $X$ be the vertex cut of $G$ with $|X| = n-3$ and let $V-X = \{x_1, x_2, x_3\}$, $X = \{v_1, v_2, ..., v_{n-3}\}$.

**Sub Case 3.1.** $\langle V-X \rangle = K_3$

Then every vertex of $V-X$ is adjacent to all the vertices of $X$. Then $\langle V-X \rangle \cup \{v_1\}$ is a double dominating set of $G$ and hence $dd(G) \leq 4$. This gives $n \leq 5$. Since $n \leq 3$ is impossible we have $n = 4$ or 5. If $n = 4$ then $G$ is isomorphic to $K_{4-e}$ which is a contradiction. If $n = 5$ then the graph $G$ has $dd(G) = 2$ or 3 which is a contradiction.

**Subcase 3.2.** $\langle V-X \rangle = K_1 \cup K_2$

Let $x_1, x_2 \in E(G)$. Then $x_3$ is adjacent to all the vertices in $X$ and $x_1, x_2$ are not adjacent to at most one vertex in $X$. If $v_i \notin N(x_1) \cup N(x_2)$ then $\langle V-X \rangle \cup \{v_i\}$ is a double dominating set of $G$ and hence $dd(G) \leq 4$. This gives $n = 5$. For this graph $\kappa(G) = 1$ which is a contradiction. So all $v_i$, either $v_i \in N(x_1)$ or $v_i \in N(x_2)$ or both. Then $\langle V-X \rangle \cup \{v_i\}$ is a double dominating set of $G$. Hence $dd(G) \leq 4$ and then $n = 4$ or 5. If $n = 4$ then $G$ is isomorphic to $C_5$ or $C_3(P_3, P_1, P_1)$. But $\kappa(e(P_3, P_1, P_1)) = 1 \neq n-3$. Hence $G$ is isomorphic to $C_5$.

**Case 4.** $dd(G) = n$ and $\kappa(G) = n-4$

Then $n-4 \leq \delta(G)$. If $\delta = n-1$ then $G$ is a complete graph which is a contradiction. If $\delta = n-2$ then $G$ is isomorphic to $K_{n-Y}$ where $Y$ is a matching in $K_n$. Then $dd(G) = 2$ or 3. Then $n = 2$ or 3 which is a contradiction to $\kappa(G) = n-4$.

Suppose $\delta(G) = n-3$. Let $X$ be the vertex cut of $G$ with $|X| = n-4$ and let $X = \{v_1, v_2, ..., v_{n-4}\}$.

Then $\langle V-X \rangle$ contains an isolated vertex then $\delta(G) \leq n-4$ which is a contradiction. Hence $\langle V-X \rangle$ is isomorphic to $K_2 \cup K_2$. Also every vertex of $V-X$ is adjacent to all the vertices of $X$. Let $x_1, x_2, x_3, x_4 \in E(G)$. Then $\{x_1, x_3, v_1\}$ is a double dominating set of $G$. Then $dd(G) \leq 3$. Hence $n \leq 3$ which is a contradiction. Thus $\delta(G) = n-4$. 


Subcase 4.1. $\langle V - X \rangle = \overline{K}_4$

Then every vertex of $V - X$ is adjacent to all the vertices in $X$. Suppose $E(\langle X \rangle) = \emptyset$. Then $|X| \leq 4$ and hence $G$ is isomorphic to $K_{s,4}$ where $s = 1,2,3,4$. If $s \neq 1$ then $dd(G) = 3$ or 4 which is a contradiction to $dd(G) = n$. Hence $G$ is isomorphic to $K_{1,4}$. Suppose $E(\langle X \rangle) = \emptyset$. If any one of the vertex in $X$ say $v_i$ is adjacent to all the vertices in $X$ and hence $dd(G) \leq 3$ which gives $n \leq 3$ which is a contradiction. Hence every vertex in $X$ is not adjacent to at least one vertex in $X$. Then $\{x_1,x_2,v_1,v_2\}$ is a double dominating set of $G$ and hence $dd(G) \leq 4$. Then $n \leq 4$ which is a contradiction to $\kappa(G) = n - 4$.

Subcase 4.2. $\langle V - X \rangle = P_3 \cup K_1$

Let $x_1$ be the isolated vertex in $\langle V - X \rangle$ and $(x_2, x_3, x_4)$ be a path. Then $x_1$ is adjacent to all the vertices in $X$ and $x_2,x_4$ are not adjacent to at most one vertex in $X$ and hence $\{x_1,x_2,x_4,v_1,v_2\}$ where $v_1 \in N(x_1) \cap X$ and $v_2 \in N(x_2) \cap X$ is a double dominating set of $G$ and hence $dd(G) \leq 5$. Thus $n = 5$. Then $G$ is isomorphic to $P_5$ or $C_4(1,0,0,0)$ or $K_5(1,1,0)$ or $(K_4 \setminus e)(1,0,0,0)$. All these graph $dd(G) \neq n$ which is contradiction.

Subcase 4.3. $\langle V - X \rangle = K_1 \cup K_1$

Let $x_1$ be the isolated vertex in $\langle V - X \rangle$ and $\{x_2,x_3,x_4\}$ be a complete graph. Then $x_1$ is adjacent to all the vertex in $X$ and $x_2,x_3,x_4$ are not adjacent to at most two vertices in $X$ and hence $\{x_1,x_2,x_3,v_1,v_2\}$ where $v_1,v_2 \in X - N(x_2 \cup x_3)$ is a double dominating set of $G$ and hence $n = 5$. All these graph $dd(G) \neq n$.

Subcase 4.4. $\langle V - X \rangle = K_2 \cup \overline{K}_2$

Let $x_1,x_2 \in E(G)$ and $x_3,x_4 \in E(\overline{G})$. Then each $x_i$, $i = 1$ or $2$ is non adjacent to at most one vertex in $X$. Each $x_j$, $j = 3$ or $4$ is adjacent to all the vertices in $X$. Then $\{x_1,x_2,x_3,x_4,v_1\}$ is a double dominating set of $G$ and hence $dd(G) \leq 5$. Then $n \leq 5$. Since $\kappa(G) = n - 4$ we have $n = 5$. Then $G$ is isomorphic to either $B(1,2)$ or $K_5(2,0,0)$. Since $dd[K_5(2,0,0)] \neq 5$ $G$ is isomorphic to $B(1,2)$.

Subcase 4.5. $\langle V - X \rangle = K_2 \cup K_2$

Let $x_1,x_2$, $x_3,x_4 \in E(G)$. Since $\delta(G) = n - 4$, each $x_i$ is non adjacent to at most one vertex in $X$. Then at most one vertex say $v_i \in X$ such that $|N(v_i) \cap (V - X)| = 1$. If all $v_i \in X$ such that $|N(v_i) \cap (V - X)| \geq 2$ then $\{x_1,x_2,x_3,x_4\}$ is a double dominating set of $G$ and hence $n = 4$ which is a contradiction. Then $|N(v_i) \cap (V - X)| = 1$ and $|N(v_i) \cap (V - X)| \geq 2$, $i \neq 1$. Then $\{v_1,x_1,x_2,x_3,x_4\}$ is a double dominating set of $G$ and hence $n = 5$ which gives a contradiction to $|N(v_i) \cap (V - X)| = 1$ and $|N(v_i) \cap (V - X)| \geq 2$. The converse is obvious.
3. Conclusion

In this paper we found an upper bound for the sum of double domination number and connectivity of graphs and characterized the corresponding extremal graphs. Similarly double domination number with other graph theoretical parameters can be considered.

REFERENCES


